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Note

Some remarks on long monochromatic cycles in edge-colored complete graphs

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ABSTRACT

In [On the circumference of a graph and its complement, Discrete Math. 309 (2009), 5891–5893], Faudree et al. conjectured that when $r \geq 3$, every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least $n/(r-1)$. We disprove this conjecture for small n and give a short proof of the following weaker but more generalized form: for $r \geq 1$, every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least n/r.

We use [1] for terminology and notation not defined here and consider only finite and simple graphs. In this note, we consider edge-colored graphs with r colors. For an r-edge-colored complete graph K_n, we are concerned with the length of the longest monochromatic cycle. For $i \in \{1, 2\}$, we regard K_i as a cycle of length i. Keeping this in mind, let $f(r, n)$ be the maximum integer k such that every r-edge-coloring of K_n contains a monochromatic cycle of length at least k. In this note, we will show that $f(r, n) \geq n/r$ for $1 \leq r \leq n$.

In [3], Faudree et al. proved that $f(2, n) = \lceil 2n/3 \rceil$. Moreover, using the affine plane of index $r+1$, they proved that $f(r, n) \leq n/(r-1)$ when $r \geq 3$ and $(r-1)^2$ divides n. (The affine plane construction that yields the upper bound on $f(r, n)$ is attributed to Andras Gyárfás.) They also proposed the following conjecture:

Conjecture A. *For $r \geq 3, f(r, n) \geq n/(r-1)$.*

However, the conjecture is not true as stated. To see this, consider the case $n = 2r$. It is well known (see e.g., [1]) that a complete graph K_{2r} can be factored into r Hamiltonian paths. Giving each path a color yields a counterexample for $n = 2r$, since each maximal monochromatic subgraph is acyclic. Thus the conjecture requires the assumption "$n > 2r$". However, even if we add this, there is a counterexample when $n = 2r + 1 = 7$. To see this, we prove the following:

Proposition 1. *For $r \geq 1$, there exists an r-edge-coloring G of the complete graph K_{2r} such that G contains r independent edges with different colors (i.e., G has a rainbow perfect matching) and each maximal monochromatic subgraph induces a Hamiltonian path.*

Proof. We can easily construct the desired r-edge-colored complete graph K_{2r} by the standard rotational zig–zag path decomposition of K_{2r} with the vertices on a circle. To see this, consider the case $r = 4$ and the complete graph on the integers modulo 8. Let P_0 be the path 0, 1, 7, 2, 6, 3, 5, 4, and for $i \in \{1, 2, 3\}$, let P_i be the path obtained from P_0 by adding i to each vertex (modulo 8). When the vertices $\{0, 1, \ldots, 7\}$ are arranged in a cycle, the P_i are just cyclic rotations of P_0. For every i, add color i to the P_i. Then we obtain the desired coloring of K_{2r}. Notice that the rainbow perfect matching is just the diameters 04, 15, 26, and 37. This construction can easily be extended to other cases. □

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Next we give the upper bound on $f(r, 2r+1)$.

Proposition 2. *For $r \geq 1$, $f(r, 2r+1) \leq 3$.*

Proof. Let G be an r-edge-coloring of K_{2r} as in Proposition 1. Then we may assume that G contains r independent edges $a_1b_1, a_2b_2, \ldots, a_rb_r$ such that a_ib_i has color i for each $1 \leq i \leq r$. We add a new vertex v to G and construct an r-edge-coloring of K_{2r+1} by the following manner: for each $1 \leq i \leq r$, assign color i to edges va_i, vb_i, respectively. Let G^* be the resulting r-edge-coloring. Then by the construction, we can easily check that any monochromatic cycle is a triangle in G^*. This implies $f(r, 2r+1) \leq 3$. □

In Proposition 2, letting $r = 3$, $n = 2r+1 = 7$, we obtain a counterexample to Conjecture A in this case. Hence, we need to set a more modest goal. Our result is the following:

Theorem 1. *For $1 \leq r \leq n$, $f(r, n) \geq n/r$. In particular, $f(r, 2r+1) = 3$.*

Proof. We use the following theorem in [2].

Theorem B. *Let c, m, n be positive integers with $3 \leq c \leq n$, and let G be a graph of order n with m edges. If G contains no cycle of length at least c, then $m \leq (c-1)(n-1)/2$.*

Let G be an r-edge-coloring of K_n, and let G_i be the maximal monochromatic spanning subgraph of G with color i for $1 \leq i \leq r$. Let m_i be the number of edges in G_i. By contradiction, we may assume that each G_i contains no cycle of length at least n/r. By Theorem B, we have $m_i \leq (\lceil n/r \rceil - 1)(n-1)/2 < n/r \cdot (n-1)/2$. Consequently, $n(n-1)/2 = \sum_{i=1}^{r} m_i < n(n-1)/2$, a contradiction.

Thus we have $f(r, n) \geq n/r$. This together with Proposition 2 shows $f(r, 2r+1) = 3$. □

In view of Propositions 1 and 2, we see that the bound "n/r" in Theorem 1 is best possible for $n \leq 2r+1$, but we do not know whether this is sharp for $n \geq 2r+2$. So we restate Conjecture A in the following form:

Problem 1. For $r \geq 2$ and $n \geq 2r+2$, determine $f(r, n)$.

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